DOCUMENT RESUME

ED 277 575 SE 047 677

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TITLE Varga: On Probability.

INSTITUTION Southern Illinois Univ., Edwardsville. Div. of Education.

PUB DATE 14 Feb 81
NOTE 51p.
PUB TYPE Guides - Classroom Use - Guides (For Teachers) (052)

EDRS PRICE MF01/PC03 Plus Postage.
DESCRIPTORS *Educational Games; Elementary Education; *Elementary School Mathematics; *Learning Activities; *Mathematics Instruction; *Probability; *Problem Solving; Resource Materials
IDENTIFIERS *Combinatorics

ABSTRACT

This booklet resulted from a 1980 visit by the author, a Hungarian mathematics educator, to the Teachers' Center Project at Southern Illinois University at Edwardsville. Included are activities and problems that make probablility concepts accessible to young children. The topics considered are: two probability games; choosing two beads; matching colors; heads and tails, real or fake; counting the throws until you get a six; variations on the theme counting the throws; and throwing colored dice. (MNS)

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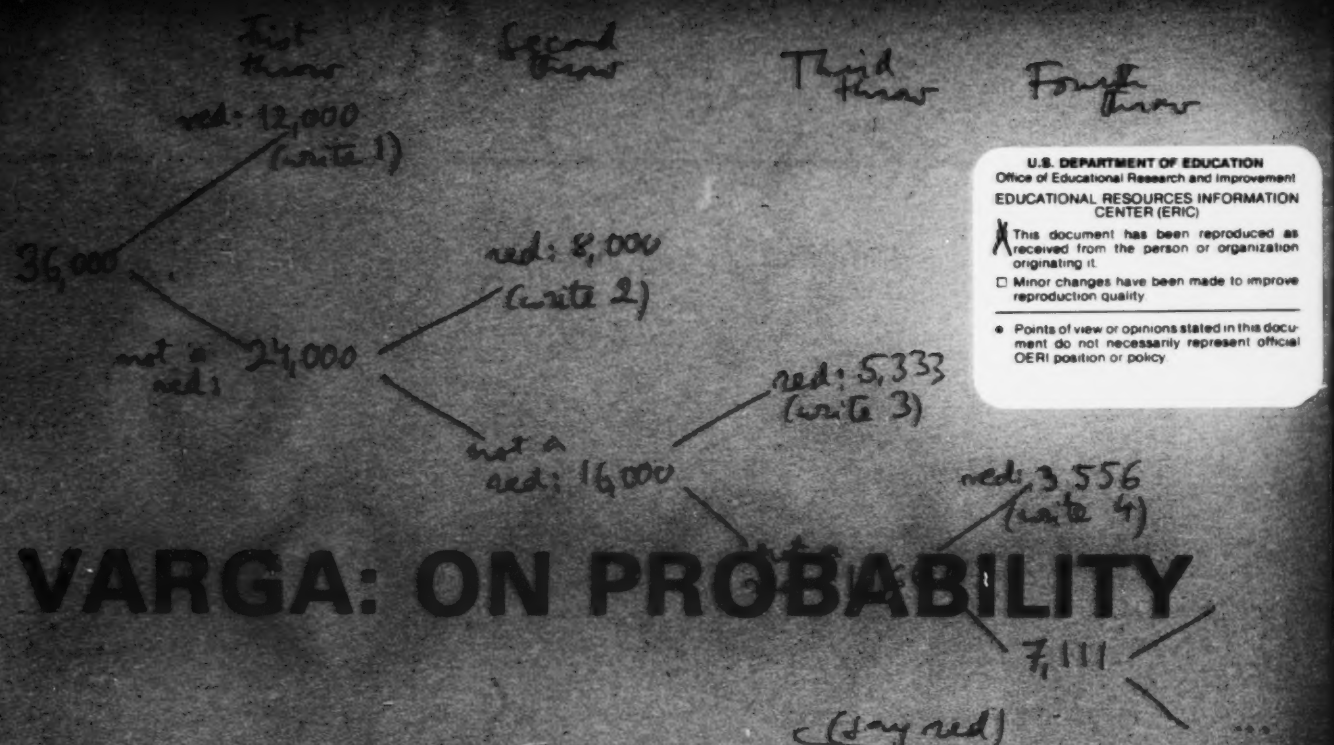

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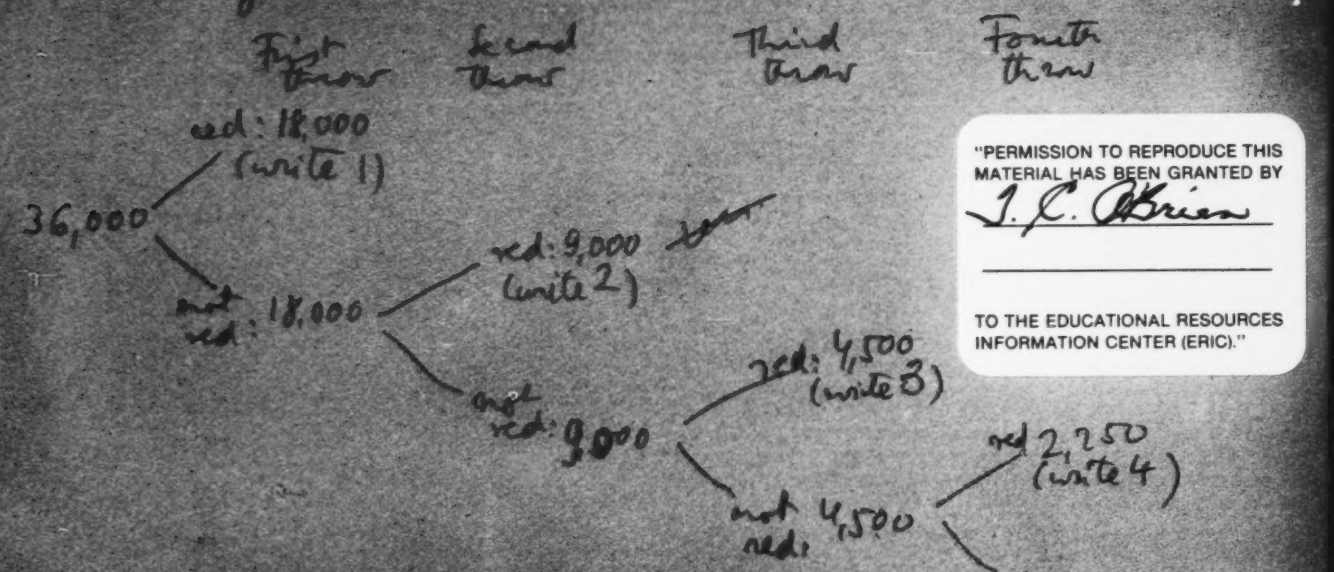
VARGA: ON PROBABILITY

If three faces have the same colour (say red), then the tree diagram will be like this:



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~~The~~ Instead of a die, a coin can be thrown. Student can then count the throws until a Head appears. Though perceptually different, the situation will be mathematically the same as with a die having three faces with the same color (this last case).

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Introduction

This book had its beginnings in the summer of 1976 when, as American representative to the retirement ceremonies for Prof. dr. Hans Freudenthal, I had the good fortune to meet Tamas Varga, who was representing Hungary. We chatted briefly and began a friendship, the high point of which was Varga's two month appointment as Visiting Scholar with the Teachers' Center Project in late 1980.

This book is one result of that visit. Additionally, we are pleased to make available three other pieces by Tamas Varga. The Mathematics of the Arch is a collection of observations, problems and investigations for children ten and up. "Varga: On Schooling", a lecture on education in general with suggestions for future trends, appears in SEEDBED 9. And "An Interview with Tamas Varga" is a videotaped interview on Varga's views and observations of American education, soon--we hope--to be available in manuscript form.

Three observations are appropriate in introducing Varga: On Probability. (Varga himself needs no introducing to Americans, I should hope.) One is that educational growth is a function of people and ideas, not big bureaucracies and corporate linearity. Call it a fire-in-the-belly approach. Varga has fire and it comes through in all his work.

The second observation comes from that 1976 meeting of Freudenthal's who in a sense is the spiritual father of Varga's Teachers' Center involvement. In his retirement address, Freudenthal said something like this: "What have I done with my career [at the University of Utrecht]? I have tried to make important ideas trivial!!!" When I heard that I thought that I had heard wrong. How could important ideas be trivial? Freudenthal

went on: "I have tried to make the big ideas of mathematics, known only to a few, known to the world, even to little children, so that the big ideas would become commonplace."

In Varga: On Probability, it's fair to say, you see big ideas made simple (or are big ideas usually simple to begin with?) so that the world, especially children, might regard them as commonplace.

The third observation is that the probability activities herein are lovely and provocative and ingenious. I know of nowhere else in American math education where probability and combinatorics are so thoughtfully treated, nor do I know of any sensible attention given to these issues with little kids. Even more important, I think, is the flavor that Varga brings to the work. Knowledge is a constructive act, he argues, not a thing to be shoved into memory barrels.

That view, which is important, needs to be trivialized in the Freudenthal sense.

Warmest thanks to Tamas Varga for his work with local teachers, for both gentleness and belly-fire, and for making the present works available to American teachers and math educators.

Thomas C. O'Brien

14 February 1981
University City, Mo.

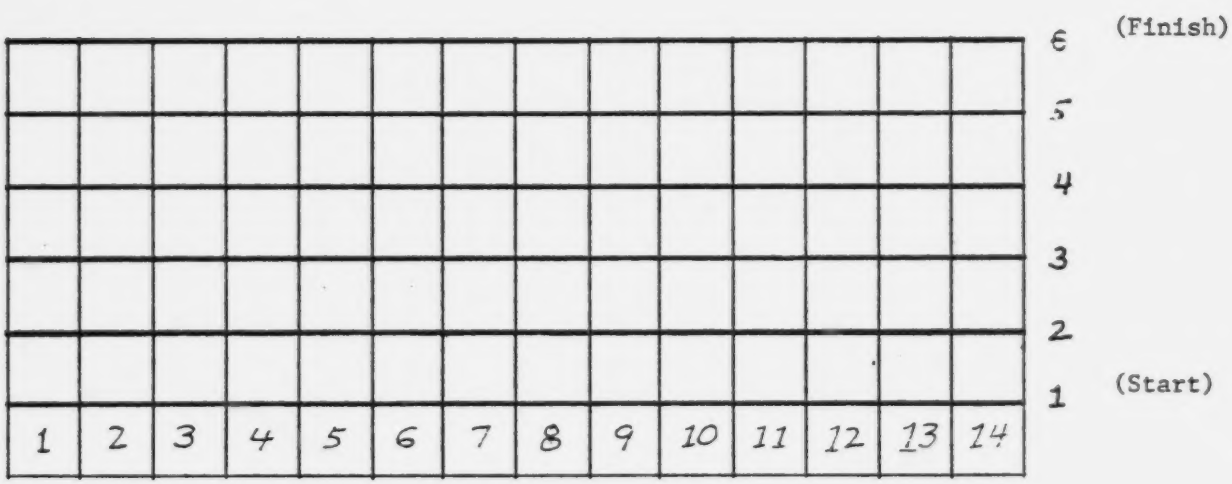
Table of Contents

Introduction	i
Two Probability Games	1
Choosing Two Beads	5
Matching Colors	14
Heads and Tails - Real or Fake?	20
Counting the Throws Until You Get a Six	26
Variations on the Theme "Counting the Throws"	37
Throwing Colored Dice	41
Epilogue	46

Two Probability Games

Game #1 "Race With Two Dice"

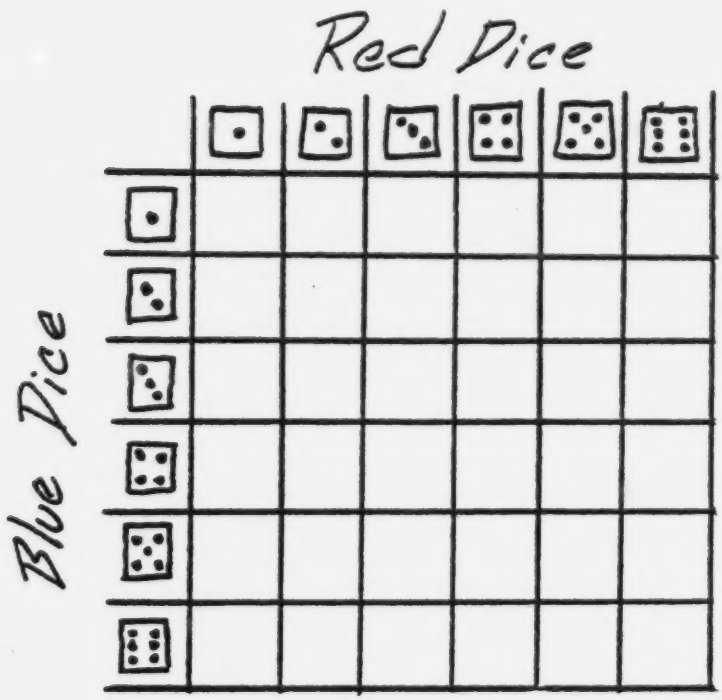
Kids choose among fourteen number cards, numbered 1-14. Two other kids who have no numbers throw two dice, add the dots and shout the result. He who has that number can make one step forward. At the beginning of the game the children are lined up behind a line. Five parallel lines show where they may step. The one who passes the front line wins.



The children may continue the game after that, for instance until two others reach the front line. Then the first gets a gold medal, the second a silver, and the third a bronze. After that they can change their numbers. They will probably want to choose the numbers winning medals. To avoid disputes more than one can choose the same number or numbers. None of them is likely to choose 1, 13, or 14. They may

even realize that these numbers have no chance for allowing even one step. Numbers like 2, 3, 11, or 12 may still appeal to them for a while, but after a certain number of games they increasingly favor the numbers in the middle such as 5, 6, 7, 8 or 9. Let them discuss this. Depending on their age they may simply accept as a fact that the numbers in the middle come up more frequently than those on the two extremes, or else they may try to find a reason why.

A further game may enhance such reasoning. This consists of throwing the two dice again, this time without letting the kids choose among the numbers, but placing the number cards on a square grid. In this game the two dice should be visibly different. One of them may be blue, the other red. The square grid looks like this:



When the two dice are thrown the kids shout the two numbers, telling which number is on the blue and which is on the red, and then their sum. For instance blue 2, red 5, 2+5=7. Then someone puts a card with the number 7 in the space which is in the row of the blue 2 and the column of the red 5. So they go on placing 5 to 8 cards on the spaces where they belong. Then they may start to discuss questions like, "Where could we put another 7?" "Another 5?" "Where will be the 4's, the 8's?" They continue filling the grid with the number cards without actually throwing the dice. They find that there are six 7's, but only five 6's, and five 8's, four 5's and also four 9's and so on up to just one 2 and one 12. The teacher does not tell them anything about this activity being related to the previous two dice races, but there will probably be kids who see the connection between the two activities. This will more likely be the case if after the second game the teacher goes back to the first game and asks, "Why are the numbers in the middle better to choose than those at the two ends?" If they still do not see the connection then it is better to let them think about it rather than give them the answer.

Game #2 "Raindrops"

The same square grid can be used as in the follow-up activity of Game 1. The 6x6 grid represents a square tile in a courtyard. A slow rain starts. The kids decide by throwing two dice where the next raindrop will fall. This time they put round markers in their respective spaces. Each marker represents a raindrop. They go on until they throw the two dice exactly 36 times so that they will have 36 raindrops in the 36 squares. But after just a few throws they are invited to

guess what the situation will be at the end. Will there be just one raindrop in each of the 36 spaces? Or will some spaces remain empty and some with two or maybe more drops? How many with no drops? How many with 1, with 2, with 3, etc.? They make their guesses and then they finish the experiment. The guesses are compared with the experiment. Many adults would believe that the most likely distribution of the raindrops is an equal one, though they certainly feel that what actually happens can be very different from that. Kids sometimes are better in their guesses than adults. What actually happens cannot, of course, be exactly predicted, but if the experiment with the 36 drops is repeated many times, then the average number of empty spaces, of spaces with 1 raindrop, with 2 raindrops, etc., shows a very clear pattern which is this:

	0 drops	1 drop	2 drops	3 drops	4 or more drops
in	13	13	7	2 spaces, on the average.	

More generally with n spaces and n raindrops in them the average number will be (if n is big enough) near to

0 drop	1 drop	2 drops	3 drops
$\frac{n}{2.7}$	$\frac{n}{2.7}$	1/2 of the previous	1/3 of the previous, etc.

In some of the cases the numbers may be somewhat different, though rarely very much different from these. The number 2.7 is only an approximation of a very famous number of higher mathematics called *e* after the German mathematician Euler, who lived about 200 years ago. A closer approximation of *e* is 2.718.

Choosing Two Beads

Let us put in a box five beads: two of one color, three of another (for instance two blue, three yellow). If we choose two of them without looking, we would come up with both blue or both yellow or mixed. Which of the three possibilities is the most likely, the least likely, in between? Or maybe some of these are equally likely?

As usual, we can estimate the probabilities based on our intuition; or we can "measure" the probability by experimenting; or else we can calculate these probabilities by theorizing the situation and building a mini-theory around it.

Whatever the case may be, there is one underlying assumption about the beads: they have to be indistinguishable to the fingers (and for that matter to the other senses), differing only in their color.

We usually invite students to write their guesses before each choice of two beads. This can happen by simply writing on a paper the initials of the two colors, such as BB or YY or BY. If their guess proves to be correct then they circle the two letters or mark the guess in some way to indicate their success. The two beads are returned to the box and mixed with the other three to restore the initial situation. While this is being done the students write another guess; they can write the same thing again if they like. Then this draw is matched with the actual outcome, etc.

After a few trials we suggest that they write a guess not just for one trial, but for several trials, maybe for ten. They don't have to predict the order of the outcomes, just their frequencies: how many will be BB, how many YY, how many BY. A convenient way to record their

predictions is to use a table like this. They should write three numbers which add up to ten.

BB	
YY	
BY	
	10

Suppose someone has predicted the numbers 2, 3 and 5 and the actual outcomes are 0, 4 and 6. How good was the prediction? We can calculate the differences and add them up.

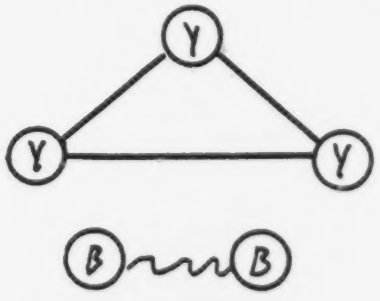
	Predictions	Outcome	Differences
BB	2	0	2
YY	3	4	1
BY	5	6	1
	10	10	4

The sum of differences is 4. In one sense, this is a measure of the quality of the predictions. If the sum of the differences is a relatively small number then the prediction was not very far from the outcome; a relatively large sum indicates a poor prediction. In this case, four indicates a very good prediction.

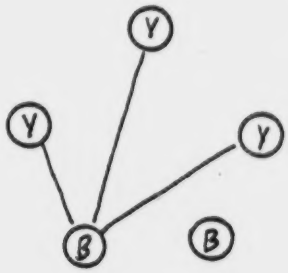
Other trials of ten may be rather different from this one. A good prediction in one group or at another time may prove to be a poor prediction in another group or another time. That is why it was suggested that the sum of the differences between the predictions and actual outcomes is a measure of the quality of the predictions in one sense. The students will realize this after a while. But, it is better to leave it up to them to discuss this question. If someone has predicted the outcome with much precision, does it necessarily mean that he is a good predictor and likely to predict the outcome very precisely again next time? Could it happen that the outcome is very much different from what can be expected and from what usually happens?

In order to see what usually happens, we would need too much time. We can't afford to make a 100x10 trials and look at the general trend. We could do that with a computer. But computers are not yet sufficiently available, nor are most teachers able to design programs simulating probability experiments. This will hopefully happen 10 or 20 years from now. (Although as this was written, I did just such simulation with a Sinclair computer. Sinclairs cost $204.95.) But even without repeating the same experiment many times we can say something about what can be expected. This we can do by what was called above a mini-theory.

This time the theory is very simple indeed. There is only one way of choosing two blue beads, but there are three ways of choosing two yellow ones.

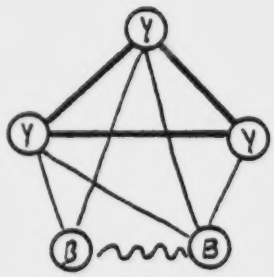


Hence YY is more likely than BB. It is actually three times more likely. How about BY? There are six ways of that since one of the B's can go with each of the three Y's.



And the other B can again go with each of the three Y's. (The students may start with one B pairing it with the two Y's or they reason it out some other way.)

The following figure summarizes the three possible outcomes and their frequencies.

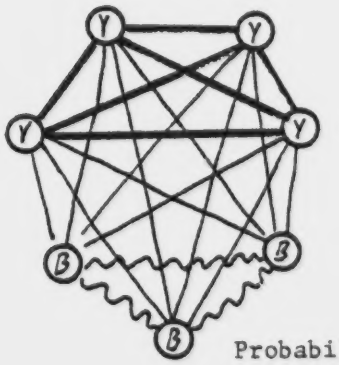


The waved line shows the single BB outcome. The heavy lines indicate the three YY outcomes. The thin lines show the six mixed outcomes. They add up to 10 (5 sides and 5 diagonals of a pentagon). Each of the 10 possibilities are equally likely; each of them has the probability .1. Therefore, the probability of BB is .1, that of YY .3, and that of BY is .6. This is exactly the same as saying that in ten trials the expected value of the number of BB's is 1, that of the YY's is 3 and that of the BY's is 6. In twenty trials these numbers are doubled.

What actually happens can be exactly this and if the ten trials are repeated many times then it is very likely that this will occur more frequently than any other number combination. But there are many other number combinations and results can vary widely. Such considerations hopefully lead the students to recognize that it is better in the long run to predict what theoretically can be expected than what actually happened during a few trials. Even if your prediction was very far from the actual outcome for ten trials, you may be a better predictor than another guy whose prediction exactly matched the outcome or was very near.

There may not be enough time to continue, but if there is unlimited time the truth eventually comes out! (This is not strictly true. What is actually true is that by increasing the number of trials there is an increasing probability that the average outcomes more and more approximate the theoretical values. This assertion can be formulated quite precisely at a higher level of mathematization.)

The number of beads can be changed and it is still easy to calculate the probabilities by connecting the letters. For instance, suppose we had four yellows and three blues and we drew two beads.



BB 3
YY 6
BY 12
21

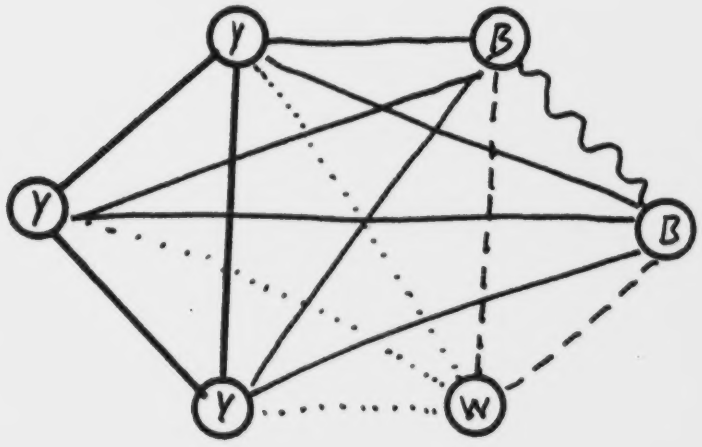
Probability of BB = $\frac{3}{21} = \frac{1}{7}$ .14 = 14%

Probability of YY = $\frac{6}{21} = \frac{2}{7}$ .29 = 29%

Probability of BY = $\frac{12}{21} = \frac{4}{7}$ .57 = 57%

This means that in twenty-one trials we can expect 3 BB's, 6 YY's and 12 BY's; in one-hundred trials we can expect approximately 14 BB's, 29 YY's and 57 BY's.

The number of colors can also be increased. For instance:



~~	BB	1
——	YY	3
——	BY	6
- - - - -	BW	2
.....	WY	3

Here we have five possibilities: three possibilities involve a mix, two involve one color (WW is missing since there is only one white). The probability of BB is the smallest at $\frac{1}{15}$. Then comes BW ($\frac{2}{15}$). The probabilities for YY and WY are equal at $\frac{3}{15}$ and BY has the greatest probability ($\frac{6}{15}$). It is again easy to calculate the probability in approximate decimals or percentages and relate them to any number of trials. Calculating the probabilities in percentages amounts to the same as calculating the expected values in one-hundred trials. This we can do by multiplying the probabilities by 100. If you want to calculate the expected values of thirty trials, then you multiply the probabilities by 30. Let's do this. (It is convenient to use a calculator.) The results rounded to the nearest integers are:

Probability			In 30 trials
.07	BB	$30 \times \frac{1}{15}$	2
.20	YY	$30 \times \frac{3}{15}$	6
.40	BY	$30 \times \frac{6}{15}$	12
.13	BW	$30 \times \frac{2}{15}$	4
.20	WY	$30 \times \frac{3}{15}$	6
			30

- -

There will (one hopes) be some children who discover that it is possible to calculate the number of mixed pairs by simple multiplication. (With 4 yellow and 3 blue beads, there were 12 mixed pairs out of a total of 24 pairs.) What about the total number of pairs? If there are many beads then the calculation can get complicated.

Let us look at the possibilities:

1 bead	0 lines	
2 beads	1 line	
3 beads	3 lines	
4 beads	6 lines	
5 beads	10 lines	
6 beads	15 lines	
7 beads	21 lines	and so on

What is the rule?

There are several ways of finding the rule and calculating the number of pairs chosen from 8, 9 or more beads. One way is to realize that the differences between the consecutive numbers are the numbers 1, 2, 3.

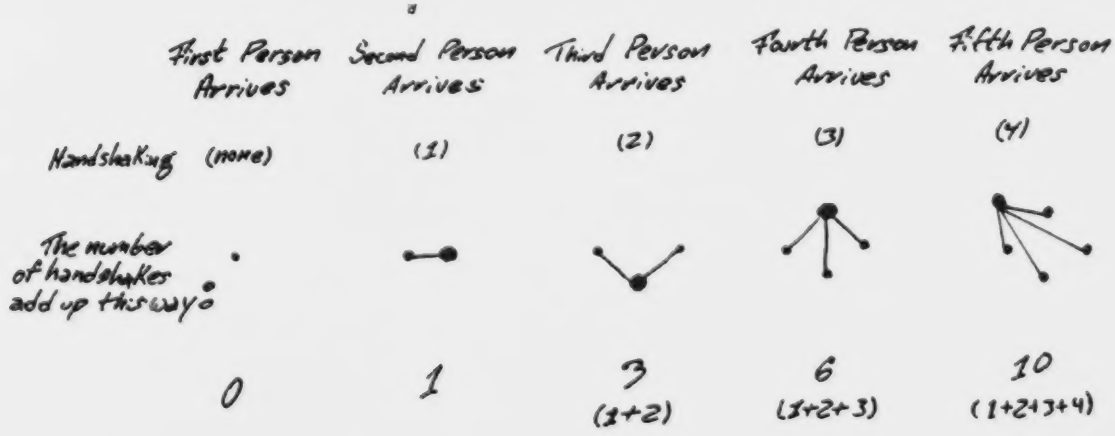
0 1 3 6 10 15 21
1 2 3 4 5 6

The pattern is in fact valid for any number of beads. How do we know that it is valid? Generalizing from a few cases is dangerous. It does not always lead to a correct conclusion. So let us try to find another approach. Let's pretend that the beads are persons participating in a tournament where everybody plays with every other player. If there are 10 of them, then each on them will have 9 adversaries, 90 in all. The connecting lines represent the matches.

Is this calculation correct? Let's apply it to 6 players. Each of them plays with 5 other persons. This gives 6×5 or 30 matches. But there are only 15 connecting lines. Where is the error?

Let us let the children discuss it and find out that the product of 6×5 (or 10×9 and so on) gives double the number of matches.

Still another way of looking at these numbers is to imagine that people come one by one to a party and each time somebody arrives everyone of those already present shakes hands with him (or her).



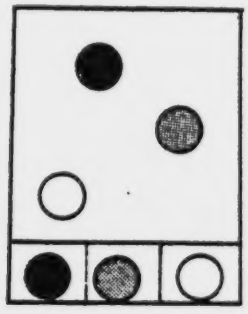
With each new arrival the number of handshakes is increased by the number of those already present, since each of them shakes hands with the person who arrives. The numbers which are produced in this way are called triangular numbers. The number of possibilities if we choose two colored beads can always be calculated by finding the appropriate triangular number. This can be done by adding up the numbers from 1 to a number one less than the number of beads of the same color (for instance 1+2 for 3 beads, 1+2+3 for 4 beads) or it can be done by multiplying two numbers and halving the product as we did above ($\frac{6\times5}{2}$ for 6 beads, $\frac{10\times9}{2}$ for 10, etc.).

It is not necessary--and may even be detrimental--to teach such rules to students. It may take more time for them to find and assimilate these rules, but if they do so they act as mathematicians and enjoy what they do. Even if some children never get beyond the level of connecting the points and counting the lines, let them do what is appropriate to their level instead of pushing them to a level which may be void of sense to them!

Though the mini-theory can easily be generalized for the case of beads with more than two <u>colors</u>, generalizing for more than two <u>choices</u> is not so easy. Leave that to more advanced students. There may be some in your class!

Matching Colors

A box with a transparent lid is divided by a wall into two equal parts. At one side of the wall 3 beads--a blue, a white and a red--are placed so that they cannot slide. Their order is fixed. On the other side of the wall 3 identical beads can move freely.



If you shake the box and let them roll against the wall, you get various results. The order of the movable beads can be the same as the order of the fixed beads so that blue matches blue, white matches white and red matches red. This does not always happen. More often only one of the beads rolls close to the bead of the same color, the other two do not match. Sometimes there is no matching at all. Kids count the matching colors: 1, 0, 1, 3,..., and they often predict that next time there would be two matchings. They are disappointed that this never happens. It takes a while until some of them realize that this can never be the case: if two colors match, then the third has to match as well. Some do not believe that and are not convinced by a verbal explanation. The best way to convince them is to let them open the box and try to place the three movable beads some way to have two matchings,

but not three. The personal experience that this is impossible is the only convincing argument for them.

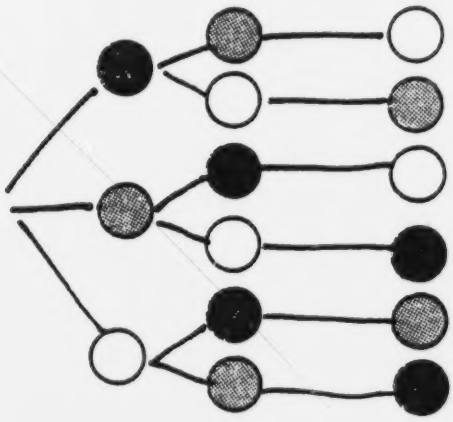
A few times we invite children to predict only what will happen next. Then we go on, as usual, to a stage where they have to predict the results of a number of trials, maybe 12 or 20 or 30. They can record the predictions and the results of the trials this way:

Number of Matchings	Predictions	Trials	
0	3	IIII	4
1	3	HHI	6
2	4		0
3	2	II	2
Sum	12		12

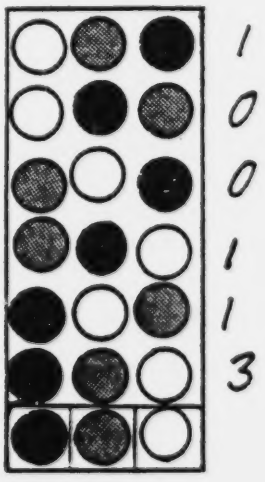
As in the case of choosing two beads, they can compare the closeness of their guesses to the experimental results by adding the differences. In this case the sum of the differences is: 1+3+4=8.

Again the closeness of the predictions to the experimental results may be a matter of chance. The experimental result may be rather different from what could be expected and the prediction may be biased in the same way. The question arises, "What can be expected?" Again we are to develop a mini-theory. First we explore all possible orders of the three movable beads. There are 6 of them. Why 6? From their experiences children distill something like this: the first bead from the left can be either blue or white or red. In each of the three possible cases the two remaining beads can occupy the two empty places

in two possible ways. This gives $3 \times 2 = 6$. Here is a tree diagram showing the six possibilities.



After having found the six cases and having convinced ourselves that there are no more cases, the next step is to compare the six possible orders, one by one, with the original order, that of the fixed beads. How many colors match in each of the six cases? Figure 4 gives these numbers. Kids can summarize the findings: there are 2 possible



orders with no matchings, 3 possible orders of one matching, and 1 order with three matches. These data can be arranged in a table like this:

	0	1	2	3	matchings
possible in	2	3	0	1	ways

Before drawing the conclusion about the probabilities we have to make sure that the three possible orders are equally likely. There is no physical law to attract or repel any of the beads, forcing them toward the middle or toward one of the two edges. The six cases are equally likely and each have the probability 1/6, just as any one of the six faces on a die. (To be sure, it is a mere coincidence that the number six is both the number of the sides of a cube and the number of possible orders of three objects.) Looking at the table, we find that the probability of three matchings is 1/6, that of one matching is 3/6 or 1/2, and that of no matchings 2/6 or 1/3.

This means that 1/6 of 12 trials can be expected to result in three matchings, 1/2 of them in one and 1/3 of them in no matching. That is to say, in the case of 12 trials:

0	1	2	3	matchings can be expected in
4	6	0	2	ways.

If the number of trials is not divisible by 6, then the expected numbers will be fractional numbers rather than integers. This does not mean, of course, that something like 2 1/2 trials with 3 matchings can occur any time. The expected number is just the theoretical average. The actual average may be different from the theoretical average, but it is rarely very much different if the number of trials is large enough.

It is possible to calculate not only the expected number of the various outcomes (3, 1 or 0 matchings), but also the expected deviations from the expected numbers, or even, still one step further, the expected deviations from the expected deviations. And so on. For a teacher it may be interesting to pursue such calculations. For the students it is more important to modify the original problem slightly, for instance using 4 different beads instead of 3. The number of possible orders is easily measurable: it is 24 (any of the four colors can be the first from the left, and the other three still have 6 possible orders.)

R	O	G	Y	
4	3	2	1	Matches
1	2	3	4	0
1	2	4	3	0
1	3	4	2	1
1	3	2	4	2
1	4	3	2	0
1	4	2	3	1
2	1	3	4	0
2	1	4	3	0
2	3	1	4	1
2	3	4	1	2
2	4	1	3	0
2	4	3	1	1
3	1	2	4	1
3	1	4	2	0
3	2	1	4	0
3	2	4	1	1
3	4	1	2	0
3	4	2	1	2
4	1	2	3	2
4	1	3	2	1
4	2	1	3	1
4	2	3	1	2
4	3	1	2	2
4	3	2	1	4

0	1	2	3	4	Matchings
9	8	6	0	1	Ways

This time it is possible to have two matchings, but three matchings are impossible. (If three beads roll against the corresponding beads then the fourth is bound to roll to its place as well.) It is interesting that 0 matching is more likely than 1 matching. With three beads 0 matching was less likely than 1 matching, but the difference is little.

How about five beads? The number of possibilities is 120 and analyzing all 120 is too time consuming. But the result of the analysis is interesting. The probability of 0 matchings and 1 matching are again very near to each other, even nearer than with four beads. This probability is very near .368. What is this magic number? This is the reciprocal of the number e mentioned in the game Raindrops. For six or more beads the approximation is still better.

Heads and Tails - Real or Fake?

If you toss a coin several times and record the results with the letters H for head, T for tail then you get a row of letters something like this:

HHHTHTTTHHHT ...

It could be like this:

TTTTTTTTTTT

or:

HHHHHHHHHHHH

But rows like the last two come up very rarely. In other words their probability is very small. Just how rarely do they come up? This is a question to deal with later on.

Is a row like

HTHTHTHTHTHT

or

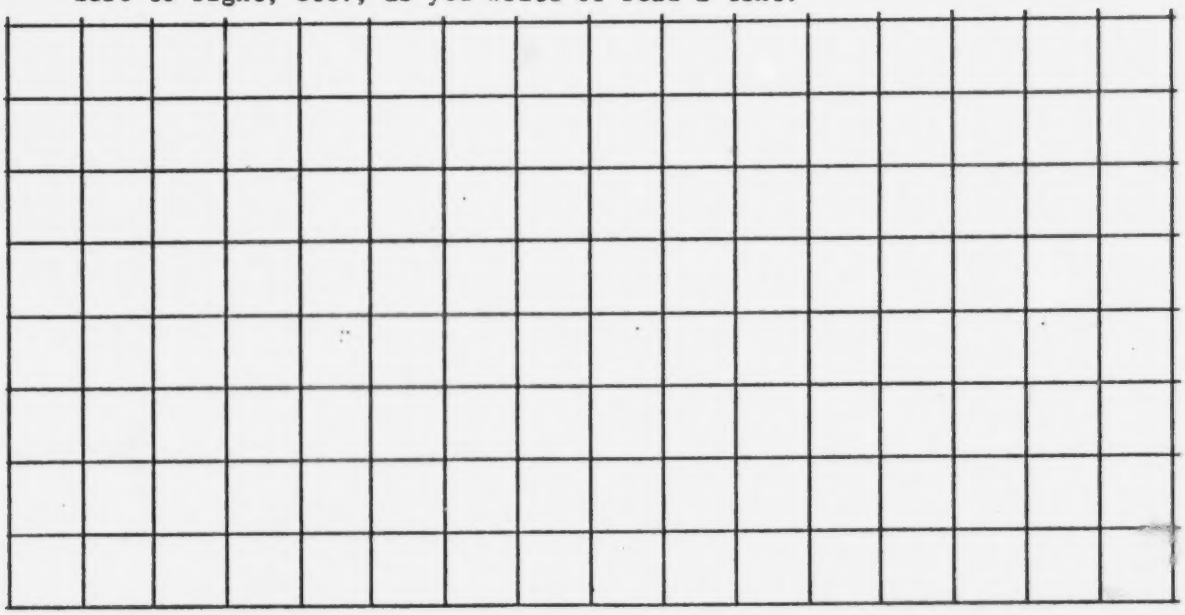
THTHTHTHTHTH

more likely, more frequent, than the former two? This is another question to deal with. Try to guess. You may be surprised.

We shall approach these sorts of questions from two different sides--through experiments and through reasoning. A third approach may precede both of these: predicting the result, both of experiments and reasoning (that is, of theoretical considerations).

One way to make a prediction is to imagine that you toss a coin without actually doing so. You record the results of the imaginary toss-ups. Try to do that. Here are 128 spaces. Fill them with H's and T's in the order first line from left to right, second line from

left to right, etc., as you write or read a text.



If you try this activity with children then you can speed it up by letting them stamp the letters with rubber stamps. It is not necessary to use the letters H and T. They can use stamps with any two different designs. They may have one stamp in their right hand and one in their left. At the end you may let them investigate whether there is some bias in favor of the left or right hand. It is important, however, that they understand that we don't expect them to stamp any regular pattern. Try to convey to them the idea by first putting the two stamps into a box and letting them choose one at random with their eyes closed, stamp with it, put it back, mix and choose one again, and so on for a while. This is the equivalent of tossing up the coins if the two stamps are not very different in size or shape.

At this point you may wonder if chance has any laws. Maybe you believe that chance is chance and law is law and they exclude each other. If by law you mean something very precise and completely regular then you are right. Yet, in a very important sense, chance events are

governed by laws. These laws are, however, only revealed gradually by increasing the number of trials and then only approximately in most of the cases.

Let us now repeat your imaginary 128 toss-ups by 128 real tosses of a coin. Fill these spaces by actually doing the experiment.

Do you think it is possible to decide which of the two recordings comes from real toss-ups and which comes from the imagined toss-ups?

Are you a very shrewd person who can imitate randomness much better than most people?

Here is a method of comparing real toss-up sequences to fake ones. You count the runs in both of them. An H is a run of length 1 if it is both preceded by and followed by a T. Two consecutive T's will make a run of length 2 if they are both preceded and followed by H's. If you count the runs of length 1, 2, etc., take care not to stop at the end of a line. Continue looking at the next line because the run may continue there. Moreover, imagine the end of the whole sequence of H's and T's tied to its beginning. For instance if all you have is this:

HHTTHT
THTTHH

then you have two runs of length 1 (H in the 1st line, another in the 2nd), three runs of length 2 (each of them is a TT, the first in line 1, the second beginning in line 1 continued in line 2, the third is in line 2), and finally a run of length 4 (HHHH, half of which is at the end of the 2nd line, the other half at the beginning of the first line.) For each run of length 1 draw a tally. Write your tallies in the following table for the imaginary toss-ups and the real ones separately. You need not record whether a run consists of 1 or more H's or 1 or more T's. Only the length of the run is important this time.

(You may have been wondering about the number 128. Why not 100, 200, or 150? The answer is that the expected number of runs of various lengths are simpler to calculate with 128 toss-ups than with 100, 200, or 150.) These are the expected numbers:

Runs	of	length	1	32
"	"	"	2	16
"	"	"	3	8
"	"	"	4	4
"	"	"	5	2
"	"	"	6 or more	2

Do you see the pattern? There are, of course, more shorter runs than longer ones. Passing from one length to the next, the number of expected runs becomes halved.

Let us check if the above really adds up to 128.

$$\begin{aligned} 32 \times 1 &= 32 \\ 16 \times 2 &= 32 \\ 8 \times 3 &= 24 \\ 4 \times 4 &= 16 \\ 2 \times 5 &= 10 \\ 2 \times 6 &= \underline{12} \\ & \quad 126 \end{aligned}$$

It is 2 less than 128, but note that if you replace the number 6 by the number 7 then the last product will be 14 and the sum of the products will become 128. Remember that the 2 in the last line is not the expected number of runs of length 6, but of length 6 or more. It could be two 7's or one 6 and one 8. Both times you get 14 in the last row.

Beware of expecting exactly this result! One hundred twenty-eight is too small a number to show a clear pattern. What you actually find in 128 real toss-ups may be rather different. But what you find when you analyze imaginary toss-ups is usually much more different, except if you are particularly good in imagining toss-ups (or if you already know much of their laws).

Why are runs of length 2 expected to be half as frequent as runs of length 1 and so forth?

Imagine that you have an H. Chances are equal that the next result will be a T or again be an H. If it is a T (and if the H was also preceded by a T) then the length of this run is 1. If another H follows the first, then all you know is that this will be a run of length 2 or more. In half of the cases it will be exactly of length 2, in the other half it will be 3 or more and so on.

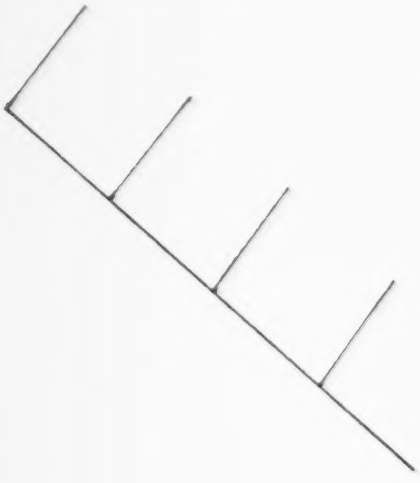
If you follow this line of thought yourself, then you will see the reason for halving the numbers as you pass from shorter to longer runs. Check in the above table that the expected number of runs of length 1 is in fact the same as the number of lengths more than 1:

$$32 = 16 + 8 + 4 + 2 + 2$$

Similarly, the number of runs of length 2 is the same as the number of lengths more than 2:

$$16 = 8 + 4 + 2 + 2$$

All this follows from the above reasoning (which is partly left to you). Can you draw a tree diagram for the situation?



Counting the Throws Until You Get a Six

In some parlour games each player can enter the game only after having thrown a 6. They throw the dice in turn and some of them succeed in throwing a 6 quite soon (without cheating), but some have to wait quite long. Kids who have experienced the excitement of waiting for a 6 are motivated to explore the mathematics behind that. Some questions which can be discussed are:

> Is it possible to throw a 6 on the first roll?
>
> Does that happen very often?
>
> Does it happen more often that you throw a 6 only on the second time?
>
> Suppose you throw the die and count the number of throws. Suppose you throw a 6 on the ninth throw. Then write the number. Now start again throwing and counting. Then you would write the number 2 if you get a 6 on the second throw, and so on. If you continue, which number do you think you will have written most frequently? The 1 (a six on the first throw), or the 2 (six for the second throw), or 3, or 4 ..., or maybe 6? Or maybe a number bigger than 6?

Let's do this in a class. A convenient way of organizing the work is for one child to throw and count 1, 2, 3 and so on, and another child to keep record and to write the number of throws needed to produce a 6. The recording may go this way:

3

6

11

1

4

After about five or ten minutes kids may learn to write the results this way:

1 //

2 ///

3 //

4

5 //

6

7 /

8

9

.....

17 /

Then the teacher collects and summarizes the results obtained by various teams in the class. One way to do this is to ask the kids, "How many ones do you have? Show me with your fingers." One "vote" for each team. The teacher adds the number of fingers shown and writes the sum on the blackboard after the number 1. Then she asks the 2's on the paper, about the 3's and about the numbers greater than 10, etc. Then comes the surprise, especially if the number of trials is quite large. The most usual winner is the 1; sometimes it is the 2 or the 3,

but numbers higher than that win the race very rarely. During five or ten minutes one can usually collect the results of several hundred tosses depending on the age and size of the class. Sometimes a lot more.

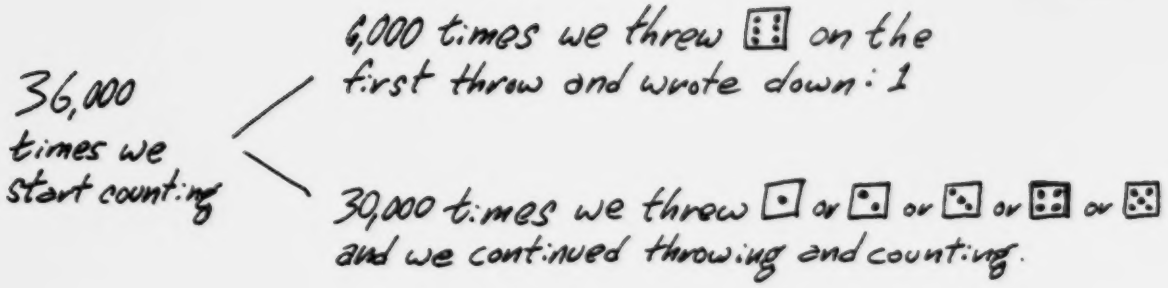
Older kids may become interested in an explanation of this phenomenon. If they made guesses before they collected the data, the guesses probably varied, between 3 and 100. Two very common guesses are 4 and 6. Students around the age of 12 are, as a rule, ready to understand the following imaginary experiment and the reasoning related to it.

"Let's pretend to throw the die until we have written down 36,000 numbers. This means that we have in our imagination thrown the die until the number 6 appeared on it 36,000 times. How many times do you think we have thrown the die?"

Hopefully they come up with an answer 6 times 36,000 or something around that. That is 216,000 throws.

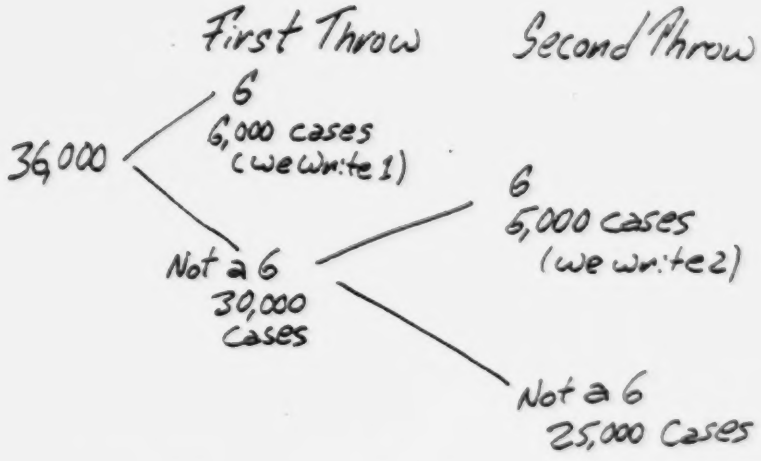
"So you have finished counting the number of throws and written down some number 36,000 times. How many times have you finished counting the number of throws after the first throw?"

If there is interaction within the class and between the class and the teacher, then it may not be necessary to ask such questions. Students may take the initiative, and with very little intervention on the part of the teacher they arrive at the idea that in about 1/6 of the cases you throw the 6 on the first roll and you write down the number 1. This happens in roughly 6000 of the cases. In the rest of them, or about 30,000 cases, you throw 1, or 2 or 3 or 4 or 5, and you continue counting!


36,000 times we start counting
6,000 times we threw ⚅ on the first throw and wrote down: 1
30,000 times we threw ⚀ or ⚁ or ⚂ or ⚃ or ⚄ and we continued throwing and counting.

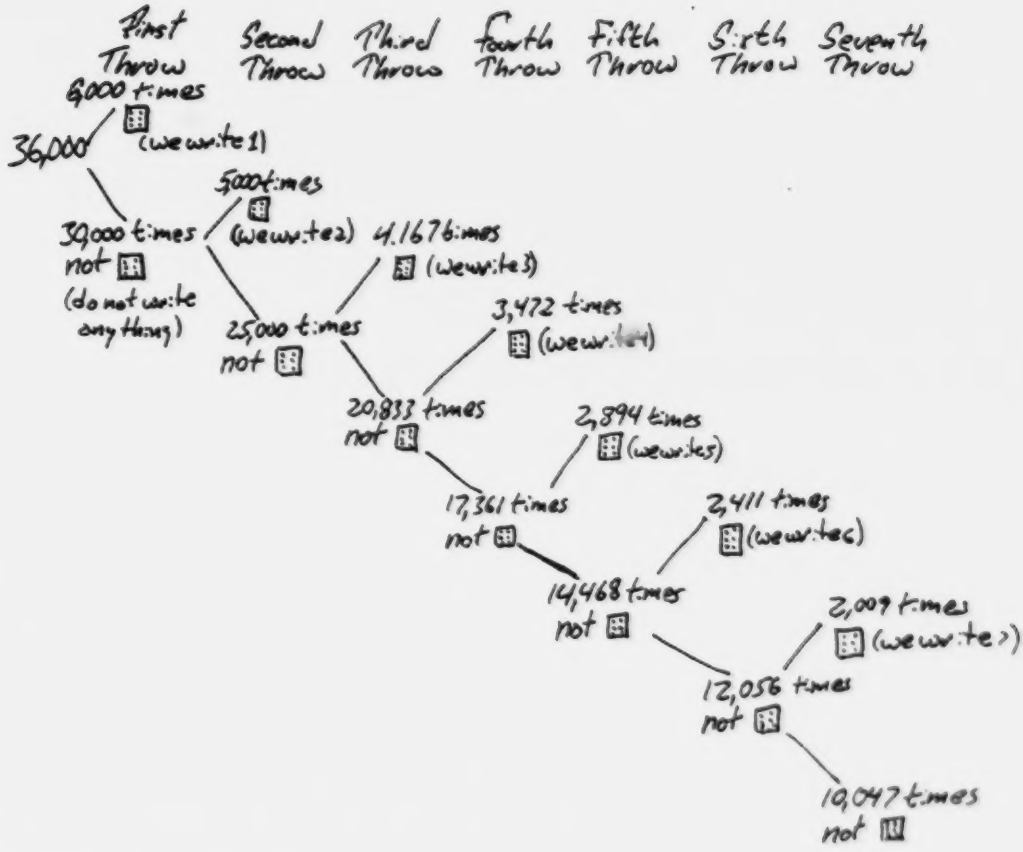
"What number will we write on the paper, if we do not throw a 6 on the first time?" When will a 6 occur? If it occurs on the second toss we write a 2, on the third toss a 3, etc. How often will the number we write down be a 2?

It may take some time and some mental effort to come up with the number 5000. The throwing is being continued after the first throw 30,000 times, and in about 1/6 of these cases the second throw will be a 6. Then we write the number 2. In about 25,000 cases we do not throw a 6 on the second time either. Our tree diagram now becomes:



"So we will have written 6000 1's and only 5000 2's. We do not yet know how many 3's, how many 4's and so on. Could you figure that out too?"

The numbers become more complicated now, but the reasoning is straightforward since it follows a well established path. Using a calculator and rounding the quotients to the nearest integer the tree diagram becomes:



So it goes on with the number of 8's, 9's, etc., shown here:

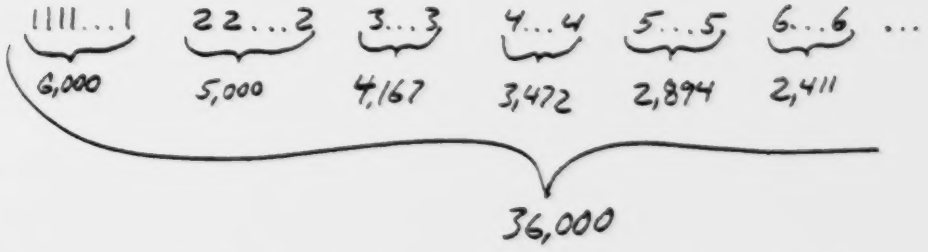
8	9	10	11	12	13	14	15	16	17
1674	1395	1162	969	808	673	561	467	389	325

18	19	20	21	22	23	24	25	26	27
270	225	188	157	130	109	91	75	63	52

28	29	30	31	32	...
44	36	30	25	21	...

It should be made clear, however, that the calculation only gives the expected numbers, or the "ideal" numbers. The actual results may be slightly different or, more rarely, very different. But the pattern is clear: the greater a number, the less likely it can be expected to appear on the paper. The factor by which these numbers decrease (6000 for the 1, 5000 for the 2, 4167 for the 3, etc.) is 5/6. Most calculators have a memory and this makes the calculation very easy.

Let's now imagine having written all these 1's and 2's and 3's in a long, long row in increasing order. We have 36,000 numbers in a row altogether:



There are several questions we can ask about these numbers. Three very important questions are:

1. Which is the most frequent number among these 36,000 numbers?
2. Which of them is the middle of the row?
3. What is their average?

The first question is very easy to answer. In fact, this was the subject of our investigation, and the surprising result was precisely that 1 is the most frequent number. This number occurs theoretically 6000 times (in practice, maybe somewhat less or somewhat more often, but rarely very much less or very much more). The number 2 is only expected to occur 5000 times, and so on.

The second question is just a little more difficult to answer. If the students do not see the way of finding the answer then intermediate questions may be, "Do you think that one of the 2's is right in the middle? Maybe the last one? If not, why not? Maybe one of the 3's? ..." Simple addition shows that the last two is only the 11,000th in the row and this is not yet in the middle; the last 3 is the 15,167th number in the row and this is still not the middle; the last 4, however, is the 18,639th number in the row which is already beyond the middle in a row of 36,000 numbers. So the middle number is among those 4's. To be very precise, there is no middle number since we have an even number (36,000) of numbers, or to put it another way, there are two middle numbers and both are 4's.

The third question may turn out to be the most difficult. First of all, students have to know what average means. Even if they learn a way of calculating an average ("add all the numbers and divide the result by the number of numbers"), such rules tend to be forgotten unless they are related to an idea, a story, or an action, and not just to the juggling of numbers.

The idea may be that three friends decide to make a trip which costs $28 per person, the money to be paid in cash. They put together whatever money they have and they divide it evenly. If each of them receives at least $28 then they can make the trip, otherwise they can not. It is the action of putting together three quantities (which is behind the idea of adding the three numbers) and sharing the sum equally (the idea behind what is called division) which has to be assimilated by them. A life situation makes the issue easier to understand. Of course they could avoid sharing and say that the money of the first and the second

and the third person put together should be enough to pay $28 for the first plus $28 for the second plus $28 for the third person. This is another, more fundamental, way of expressing what average means. If you have three numbers, then looking for their average means looking for a number which gives the same sum if each of the three numbers in the addition is being replaced by that particular number. This idea of average is more fundamental than the addition and division algorithm and it can be extended to other kinds of means such as geometric or harmonic means.

So let us suppose that by now the question is clear that we are looking for the average of 36,000 numbers. We try to add all of them by saying that the sum of the first 6,000 is exactly 6,000, the sum of the next 5,000 is 10,000 (which makes 16,000 altogether) and so on. The calculation is tedious. But even apart from that, can we ever finish it? Which is the greatest number in the row? How many numbers should we multiply and then add the results before we get the precise sum of all those numbers in the row?

The task seems to be hopelessly difficult.

It is difficult unless students take some effort to understand what they do! If we write down a number such as 2 or 11, what do these numbers represent? Well, of course, they represent having thrown a die 2 times or 11 times and having seen 6 to appear at the last throw. So the total number of these numbers is the number of throws. We already calculated that number to be 216,000. This has to be divided by 36,000, the number of numbers we wrote down. The result is 6.

The three questions we had asked and then answered one by one are related to three important quantities which are called:

1. mode,
2. median and
3. mean

We found them to be, respectively, 1, 4, and 6.

Let's do another example with these numbers:

2, 2, 2, 3, 4, 6, 6, 6, 6, 11

the mode is 6 because there are four 6's and less than four of any other number; the median is 5 because the two numbers in the middle are 4 and 6, and their mean is 5 (the number precisely between the two); and the mean of the ten numbers is 4.8 because 2+2+2+3+4+6+6+6+6+11=48 and 48÷10=4.8.

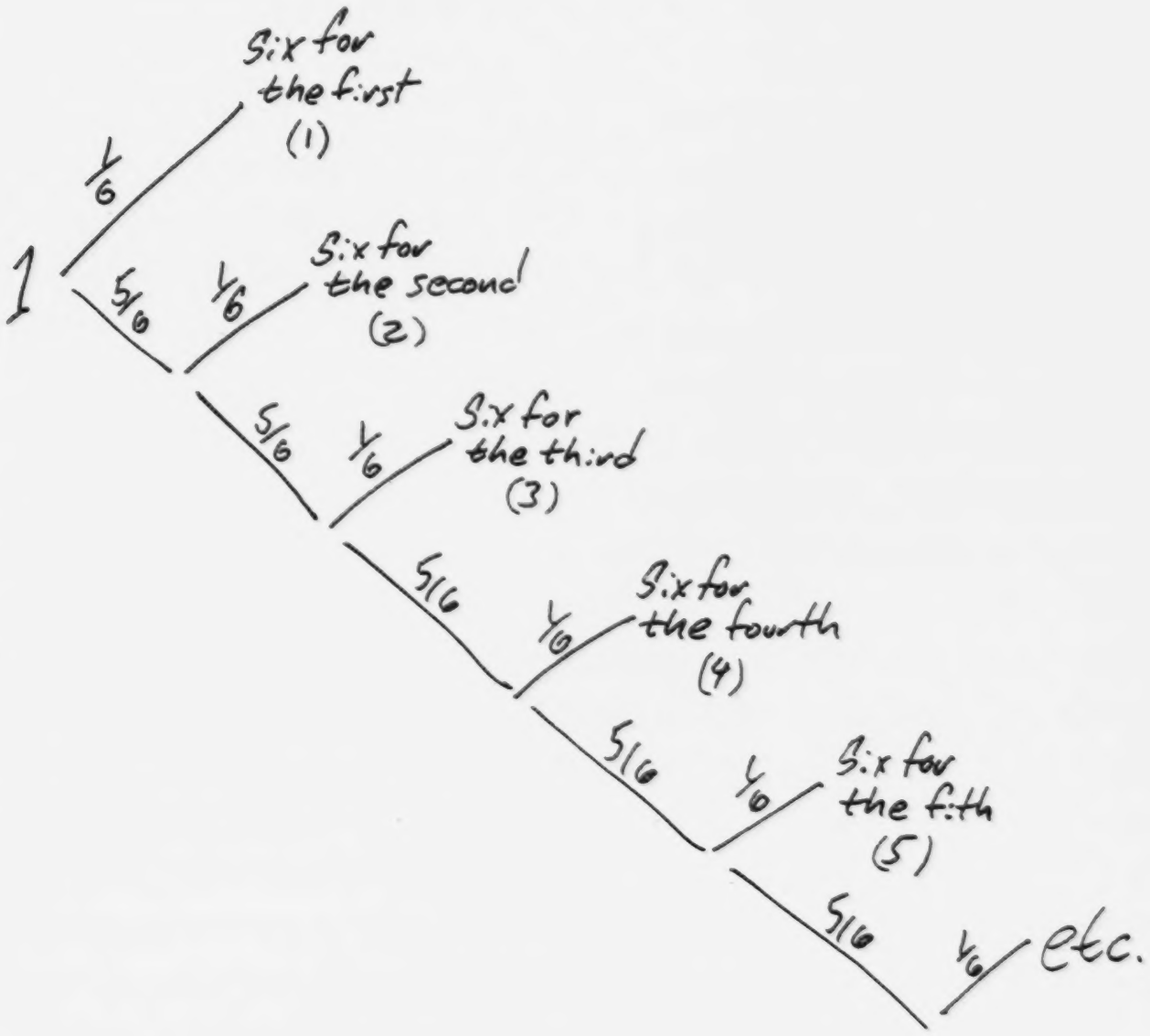
For the numbers

1.2, 2.7, 10.1, 20, and 100

the concept of a mode is void of sense (though formally we could say that each of these numbers is a mode), the median is 10.1 and the mean is 26.8.

If the numbers are values of the measurement of some quantity then they usually show a pattern which is called normal distribution. In this case there are fewer numbers in the extremes (very low or very high values). Most of them are in the middle range and the values of the mode, the median and the mean theoretically coincide. Actually they may not coincide, but they usually are very near to each other. Words which are used in the everyday language related to these quantities usually do not make a clear distinction between these three "measures of central tendency", precisely because they are often indistinguishable in practice. But it is important to make a conceptual distinction between them. (The activity described above serves this purpose because the mode, the median and the mean are well apart here.)

So far our tree diagrams started with a large number of imaginary trials and along the branches of the tree this number was sub-divided according to the various cases. On a mature level the starting number can always be the number 1 and the numbers along the branches will be fractions. For instance the tree diagram about counting the throws until a 6 appears becomes:



What fraction of the trials will result in, say, not a six for the first throw but a six for the second? The answer is $\frac{1}{6}$ of $\frac{5}{6}$. If we had 36000 trials, $\frac{1}{6} \times \frac{5}{6} \times 36000$ (5000, as we found before) would theoretically yield a six on the second roll. More generally, the compound probabilities can be calculated by multiplying the simple probabilities along the branches. The probability of throwing a six on the fourth throw is $p = \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$. The first three factors mean "not yet a six"; the last factor accounts for throwing a six on the fourth roll.

Variations on the Theme "Counting the Throws"

Younger students may not take for granted that counting until a 1 appears on a die is tantamount to counting until a 6 appears on top. Some of them even try to rationalize their beliefs saying that it is more difficult to throw a 6 than a 1 or, more generally, that it is more difficult to throw a bigger number than a smaller one. A good teacher never discards such beliefs. On the contrary, whatever the kids say should be taken seriously without, however, giving validity to their beliefs. That is the way of bringing them up to be independent thinkers rather than making them blindly believe whatever they are told. Training them to doubt is one important element of training them to think.

If children express a strong feeling that throwing a 6 is more difficult than throwing a 1 then they can either check their belief by experimentation or they can continue reasoning about other similar imaginary situations and arrive at some conclusion. Or they can do both. The best thing is to do both since experiments alone may even confirm a false, superstitious belief. (Children may find in a particular 20 rolls, for example, they have to wait longer for a 6 than for a 1. They can, of course just as well experience the contrary. In both cases their conclusion may be false.) They may not be bright enough to understand that whatever happens has nothing to do with the numbers 1 and 6. The differences are due only to chance fluctuations.

So let us turn to reasoning. Take a die and change the numbers on its faces by sticking small squares on them with dots or written numerals. If we stick a 1 over the 6 and a 6 over the 1, will that change the situation? Will the die then behave or conform to the dots below

the sticker? How about dice with colors instead of numbers on their faces? Which color is more likely to appear on the top? For which color have we to wait longer until it appears? This kind of reasoning opens up the students' minds, especially if a vivid discussion takes place which makes them more involved than would mere formal question and answer sessions.

All of the above is related to changing the perceptual variables of the situation. Apart from that, let us change the mathematical variables too.* Instead of throwing cubical die, a tetrahedral die can be used with the digits 1, 2, 3, 4 on its faces, or an octahedral die with eight different digits, or an icosahedral die with the digits 0 through 9, opposite faces bearing the same digits. (Such sets of dice are commercially available.) Even an ordinary die can be transformed by stickers to have two colors on its faces, one of them on two faces and the other on four faces. (For example, two red faces and four white faces.) The simplest thing is to take a blank cube, such as a white Cuisenaire rod, and mark two of its faces.

This is a mathematically different situation than waiting until a 6 or a 1 arrives, but the structure is the same. (If the die has six different colors on its faces then the situation is the same as with six different digits.) If there are two faces with the same color then the probability of having that particular color on the top will be 1/3 instead of 1/6. The structure of the situation is the same but the numbers are different. We can build a tree diagram similar to the one in the activity "Counting Until a Six Appears", this time with the following numbers:

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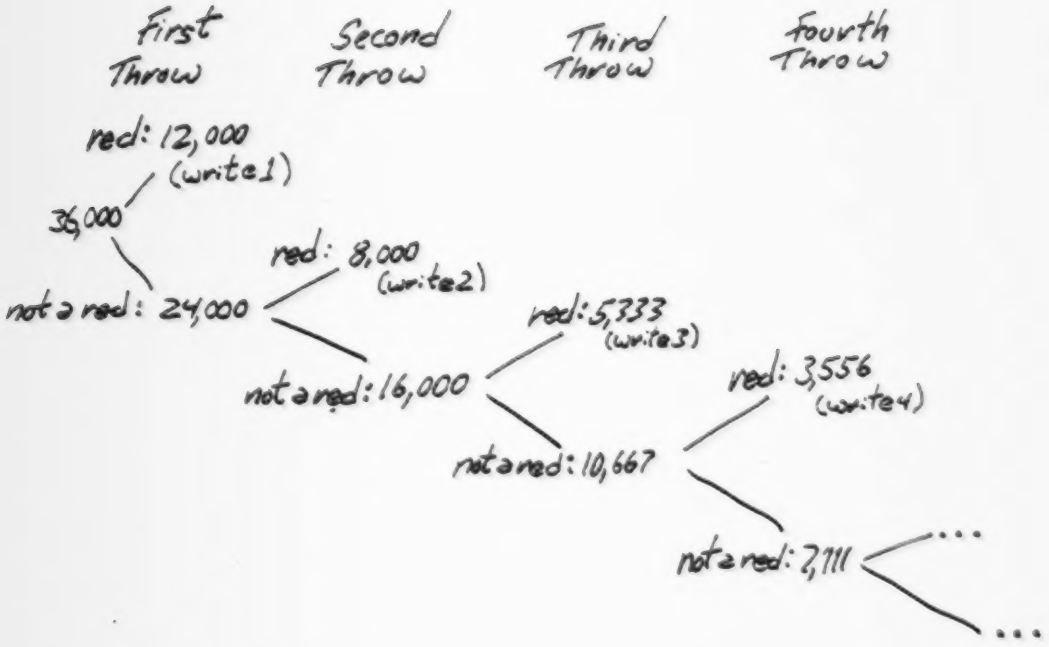
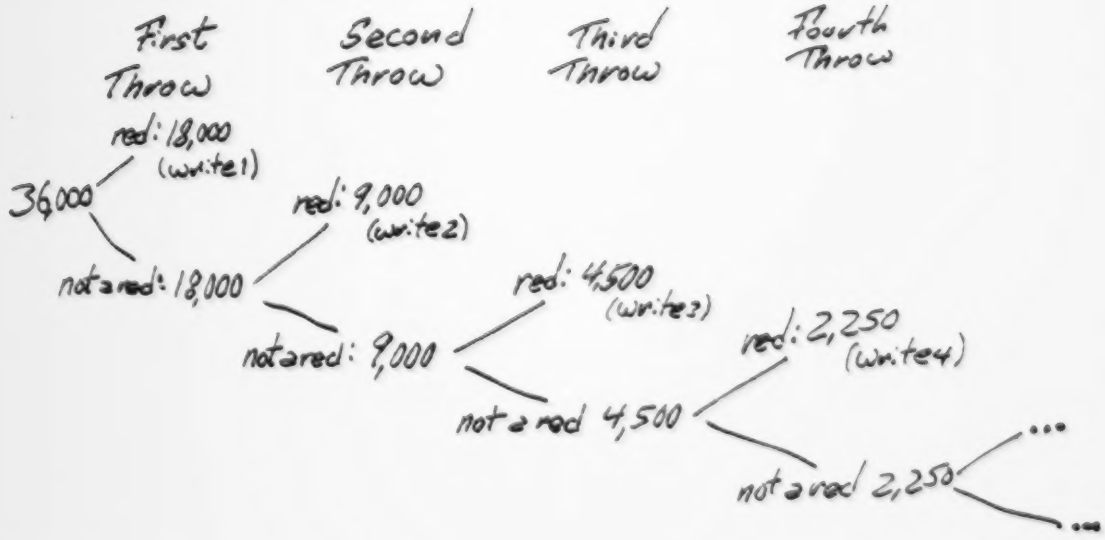
*See Dienes, Z.P., Building Up Mathematics. (4th ed). London: Hutchinson Educational. 1971.



If three faces have the same color (say red), then the tree diagram will be like this:



Instead of a die, a coin can be thrown. Students can then count the throws until a head arrives. Though perceptually different, the situation will be mathematically the same as with a die having three faces with the same color (this last case). By the way, if a coin is thrown until a head appears and the number of tosses needed to see a head is written down, and if this is done many, many times so that lots of numbers are written down, what is the mode (the most common) of these numbers? The median (the middle)? The mean?

Throwing Colored Dice

Dice with colors instead of dots are more attractive. They have the additional advantage that when you think or speak of them, then the number of throws (and the data in general) are not confused with the numbers which appear on the die. Too many kinds of numbers are confusing. Whenever possible, let's replace numbers with something else; colors are a good choice.

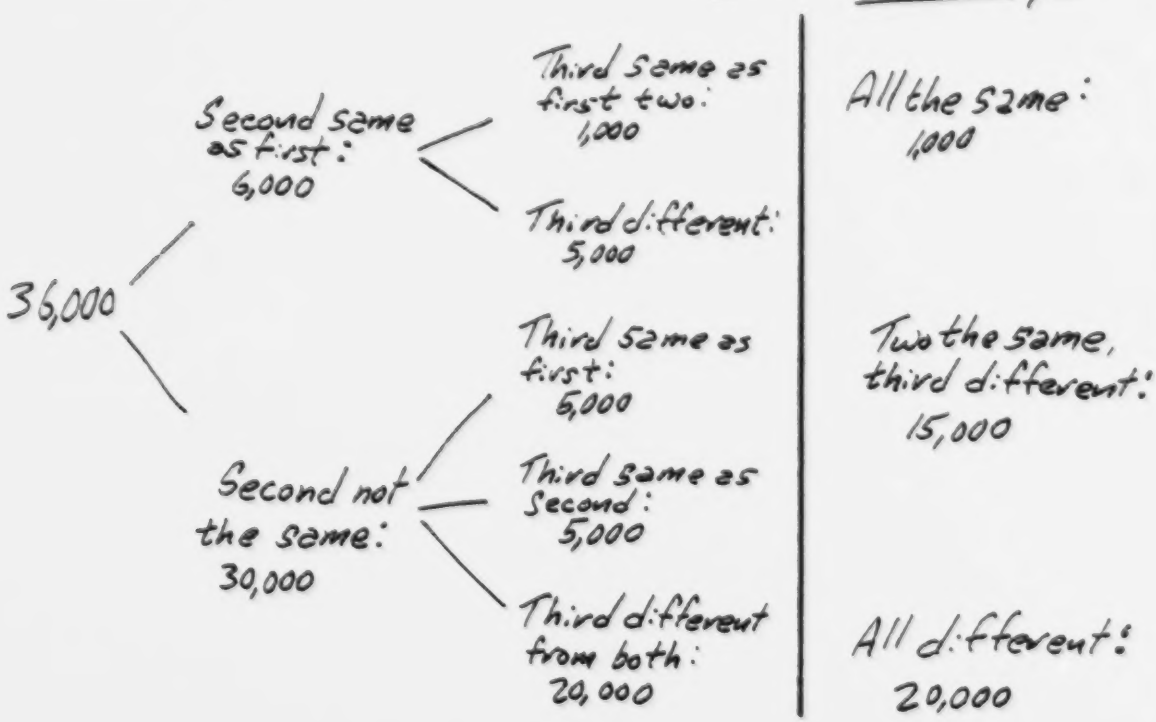
So let us have two dice with six different colors on their faces, the same colors on each die. Let us call the colors white, blue, yellow, red, green and violet. In alphabetic order: B,G,R,V,W,Y. If we throw two such dice then one of the possible questions is whether the two colors which appear on the top will be the same or different. This is an easy question. We can reason this way: Imagine that one of the two dice landed with blue on top. The second die is still in the air. There is 1/6 chance that the second will also have blue on the top. The chances are 5/6 that it will have another color. The same reasoning is valid whatever the color is on the top of the die which lands first. So if you repeat the experiment--throwing two dice many times--then in about 1/6 of the cases they will have the same color and in the rest of the cases they will have another color. In 60 throws, 10 can be expected to be "same", and 50 can be expected to be "mixed". In 100 throws these numbers are, rounded to the nearest integer, 17 and 83. It is often convenient to relate the outcomes to exactly 100 trials even if the results are not whole numbers, because then the numbers are the probabilities in percentages. In our case the probability of "same" is 17%, the probability of "mixed" is 83%.

Let us now take a third die with the same colors. We can again distinguish the two possible cases, "same" and "mixed". This time it is much less likely that the three colors will be the same. We can simply continue the previous reasoning, saying that even if the second die shows the same color as the first, (which happens in 1/6 of all throws) in only 1/6 of <u>these</u> cases will the third die again be the same color on top. This means that "same" happens in 1/36 of the throws and "mixed" happens in 35/36 of them. (Their probabilities, rounded to the nearest integer, are 3% and 97%.)

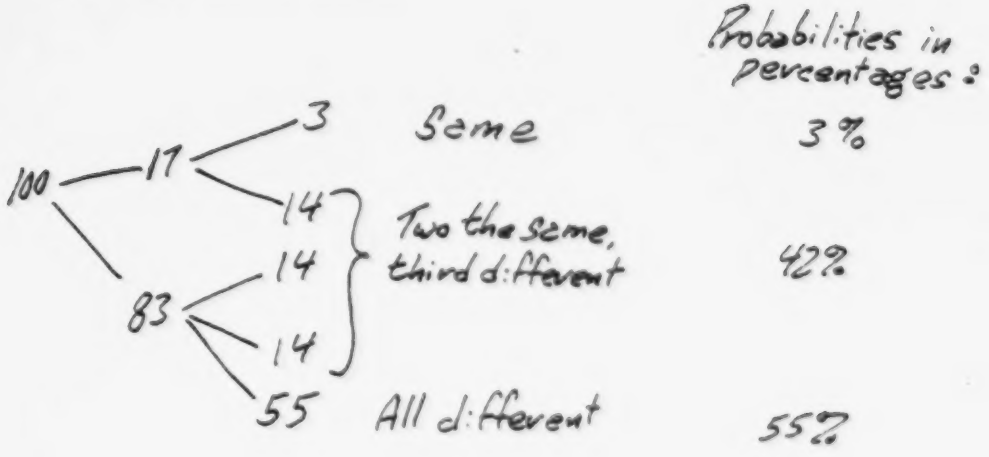
Now let us distinguish two sub-cases of the "mixed" cases: two colors may be the same or all three may be different. Which is more likely?

The following tree diagram is helpful in calculating the probabilities:

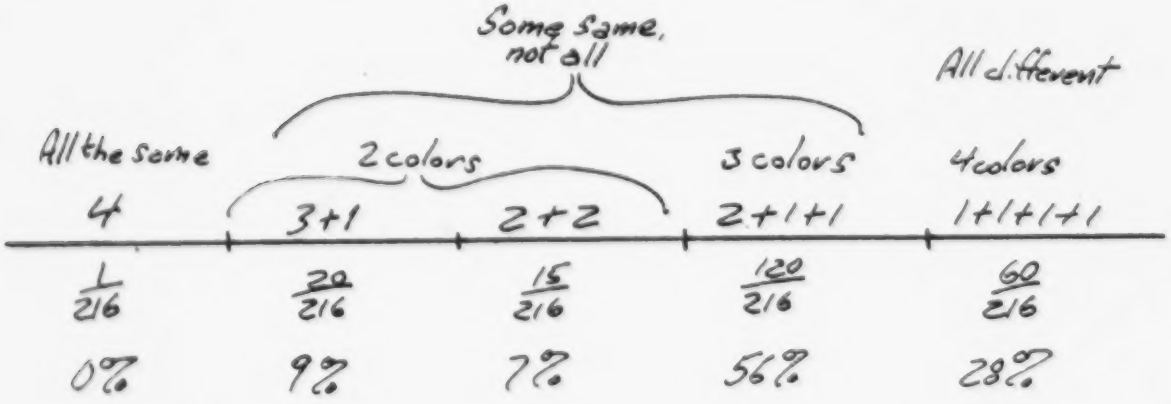
<u>Tree Diagram: Three Dice, Each B, G, R, V, W, Y</u>


Summary
Second same as first: 6,000
Third same as first two: 1,000
Third different: 5,000
All the same: 1,000
36,000
Third same as first: 6,000
Second not the same: 30,000
Third same as second: 5,000
Third different from both: 20,000
Two the same, third different: 15,000
All different: 20,000

Theoretically, in 100 throws:



From this tree diagram it is possible to calculate the probabilities of the 5 (or 4 or 3) cases we have distinguished. They are:



0% (rounded to the nearest integer) means that the probability is less that 0.5%; in fact, $\frac{1}{200}$ would be half of $\frac{1}{100}$ or one half percent and $\frac{1}{210}$ is less than that.

For 5 or more dice the calculation is so complicated that it may be more convenient to simulate the experiment, possibly by a computer, and check the guesses by analyzing the outcomes. Even the listing of the possible cases may be a tedious task for some students; for others the challenge lies precisely in the difficulty.

Now let us throw four colored dice. What are the possible cases? We can distinguish, as with the three dice, the three cases:

All the same

Some the same, but not all

All different

Another way of distinguishing the cases is this:

One color (this means that all are of the same color)

Two colors
Three colors } these are two sub-cases of some being the same but not all

Four colors

After throwing the four dice several times students will probably arrive at the above distinctions. Maybe they will come to this one, which is still more detailed:

Four of a color (this again means that all are of the same color)

Three of one color, one different

Two of one color, two of another

Two of one color, one another color, one a third color (three colors)

One each of four different colors

Whichever distinction they arrive at, they have to struggle to grasp the concepts involved. At a certain point it may be helpful to invent a notation. This is simple if we only ask about the number of colors which

appeared. More complicated is a notation adequate to the distinction described in the last listing. One way to do that is this:

4 (meaning: 4 of a color)

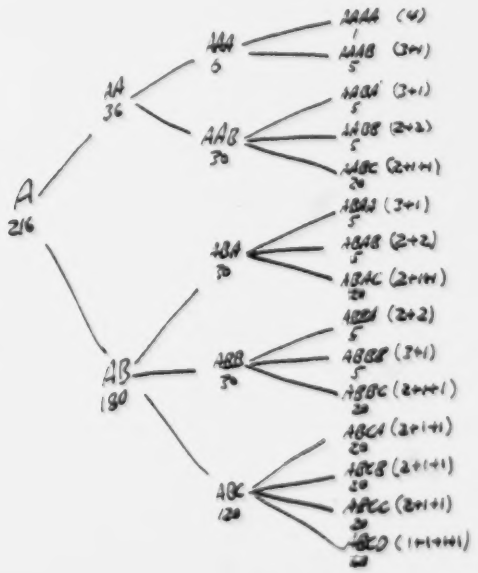
3 + 1

2 + 2

2 + 1 + 1

1 + 1 + 1 + 1

All of the above is combinatorics; systematic listing of possible cases. The next question is which of the three, four or five cases is more likely, which is less likely, which is the most likely, the least and so on. Here again combinatorics is in the background since we have equally likely cases. For each die and each color, and probabilities are measured by the number of cases and their combinations. But the reasoning becomes more and more complicated. Again it is convenient to start with an imaginary number of trials and construct a tree diagram. To avoid fractions, let us start with the number 6×6×6=216. As a further device, let us use the letter A to denote the first color to appear, B the second (if it is different from the first) and so on. Then the tree will become:



Epilogue

I hope that you have enjoyed my probability questions and problems. They are fun, but they are also demanding and productive.

Educators are asking the question, "What is basic in the education of children?" I submit that the activities you have been doing are basic. In solving the problems, you did some or all (and maybe more than) the following: You built a theory. You tested the theory. Sometimes you altered the theory in the light of reality. That's basic. You did combinatorics. That's basic. You kept records of your results. That's basic. And you dealt with chance. Chance is basic in mathematics, in industry, and in life.

My warmest regards to the Teachers' Center Project of Southern Illinois University at Edwardsville for enabling me to make available my ideas.

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